

Some results on gaps

Zoran Spasojević

Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA

(Received 30 July 1992; revised 11 December 1992)

Abstract

In his article in “Open Problems in Topology” Nyikos asked if it is consistent with $\mathfrak{t} = \omega_2$ that every \subset^* -increasing ω_1 -sequence of subsets of ω is the bottom half of some tight (ω_1, ω_2) -gap. He also asked if this can be generalized to higher cardinals and if in addition it is also possible to get $\mathfrak{b} < \mathfrak{c}$. Another question from the same article is whether $\mathfrak{p} = \omega_1$ implies that there is a tight (ω_1, ω_1) -gap in ${}^\omega\omega$. Positive answers to the second and third questions are presented here. Theorem 2.12 answers the second question from which the answer to the first question follows as a special case. I also prove Theorems 3.3 and 3.4 which can also be seen as refinements of Hausdorff’s theorem on (ω_1, ω_1) -gaps when other assumptions are made in addition to ZFC.

Regarding the first question of Nyikos, M. Rabus independently constructed a model of ZFC in which $\mathfrak{c} = \mathfrak{t} = \omega_2$ and every \subset^* -increasing ω_1 -sequence of subsets of ω is the bottom half of some tight (ω_1, ω_2) -gap. But he has not answered the generalized version of the question.

Preserving tight gaps through iteration is more difficult than just preserving gaps, and actually involves preserving a condition stronger than tightness, which in turn implies tightness. Using a completely different method from Rabus, I prove Theorem 2.12 answering in full the most general form of Nyikos’ second question and also show that it is possible to get MA to hold below a certain cardinal and at the same time preserve tightness in gaps.

Key words: Tight gaps; Strong gaps; $(\mathcal{P}(\omega), \subset^*)$; Iterated forcing

AMS (MOS) Subj. Class.: 03E35

1. Introduction

The notation and terminology are adapted from [6]. $\mathcal{P}(A)$ denotes the power set of A and for $A, B \subseteq \omega$, $A \subset^* B$ stands for $|A \setminus B| < \omega$ and $|B \setminus A| = \omega$. If A and B are both infinite then $A \perp B$ stands for $|A \cap B| < \omega$. For a cardinal λ , $[A]^{<\lambda} = \{X \subseteq A: |X| < \lambda\}$ and $[A]^\lambda = \{X \subseteq A: |X| = \lambda\}$. The same symbol, “ \perp ”,

will denote that two elements of a partial order \mathbb{P} are incompatible. Compatibility of $p, q \in \mathbb{P}$ will be denoted by $p \perp q$. This dual use of the symbol “ \perp ” will cause no confusion since it will be clear from the context which way it is being used. A family a of infinite subsets of ω is said to have the strong finite intersection property (s.f.i.p.) if any finite subfamily has an infinite intersection. Then $\mathfrak{p} = \min\{|a|: a \text{ is a subfamily of } [\omega]^\omega \text{ with the s.f.i.p. but } \neg(\exists c \in [\omega]^\omega)(\forall x \in a)(c \subset^* x)\}$ and $\mathfrak{t} = \min\{|a|: a \text{ is a subfamily of } [\omega]^\omega \text{ well ordered by } \subset^* \text{ such that } \neg(\exists c \in [\omega]^\omega)(\forall x \in a)(c \perp x)\}$. Let ${}^\omega\omega$ denote the set of all functions $f: \omega \rightarrow \omega$ and define the partial order \leq^* on ${}^\omega\omega$ by $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n < \omega$. Then $\mathfrak{b} = \min\{|a|: a \text{ is an unbounded subfamily of } ({}^\omega\omega, \leq^*)\}$. Since in this paper I will be dealing with gaps I would like to formulate \mathfrak{b} and \mathfrak{t} in terms of gaps. It follows fairly easily from the definition of \mathfrak{t} that \mathfrak{t} is also the least cardinal κ such that there exists a $(1, \kappa)$ -gap. In the case of \mathfrak{b} , Hausdorff [3] and Rothberger [10] discovered independently that \mathfrak{b} is also the least cardinal κ such that there exists an (ω, κ) -gap. Other equivalent formulations of cardinals \mathfrak{p} , \mathfrak{t} and \mathfrak{b} are given in [12].

Definition 1.1. Let λ and κ be ordinals. A (λ, κ) -pre-gap, in $\mathcal{P}(\omega)$, is a pair $\langle a, b \rangle$ where $a = \langle a_\xi: \xi < \lambda \rangle$ and $b = \langle b_\eta: \eta < \kappa \rangle$ are \subset^* -totally ordered subsets of $\mathcal{P}(\omega)$ of cofinality λ and coinitality κ , respectively, such that $(\forall \xi < \lambda)(\forall \eta < \kappa)(a_\xi \subset^* b_\eta)$. $\langle a, b \rangle$ forms a (λ, κ) -gap iff $\neg(\exists c \in [\omega]^\omega)(\forall \xi < \lambda)(\forall \eta < \kappa)(a_\xi \subset^* c \subset^* b_\eta)$. A gap is tight iff $\neg(\exists c \in [\omega]^\omega)(\forall \xi < \lambda)(\forall \eta < \kappa)(a_\xi \perp c \wedge c \subset^* b_\eta)$. A gap is strong iff $\neg(\exists c \in [\omega]^\omega)(\forall \xi < \lambda)(\forall \eta < \kappa)(|c \setminus a_\xi| = \omega \wedge c \subset^* b_\eta)$.

Clearly, a strong gap is also tight.

Definition 1.2. Let λ and κ be regular cardinals. Then $g(\lambda, \kappa)$ denotes the statement that for every \subset^* -increasing λ -sequence $a \subseteq \mathcal{P}(\omega)$ there is a \subset^* -decreasing κ -sequence $b \subseteq \mathcal{P}(\omega)$ such that $\langle a, b \rangle$ forms a (λ, κ) -gap. $sg(\lambda, \kappa)$ denotes a similar statement, but the gaps are strong. $g(< \kappa, \kappa)$ is the statement that for all regular $\lambda < \kappa$, $g(\lambda, \kappa)$ holds. Similarly for $sg(< \kappa, \kappa)$.

In the above definition, when $\lambda = 1$, I must assume that $|\omega \setminus a_0| = \omega$. I will continue to make this assumption throughout this paper without specifically stating it each time.

2.

With the necessary notation and definitions established I now proceed to answer the second question by Nyikos which will be the main topic of this section.

Definition 2.1. Let λ and κ be ordinals and $a = \langle a_\xi: \xi < \lambda \rangle$ an \subset^* -increasing λ -sequence in $\mathcal{P}(\omega)$.

$$\mathbb{P}_{a,\kappa} = \{ \langle x, y, n, f \rangle: x \in [\lambda]^{<\omega} \wedge y \in [\kappa]^{<\omega} \wedge n < \omega \wedge f: y \rightarrow \mathcal{P}(n) \}$$

with $\langle x_2, y_2, n_2, f_2 \rangle \leq \langle x_1, y_1, n_1, f_1 \rangle$ iff

- (i) $x_1 \subseteq x_2, y_1 \subseteq y_2, n_1 \leq n_2$,
- (ii) $(\forall \xi \in y_1)[f_1(\xi) = f_2(\xi) \cap n_1]$,
- (iii) $(\forall \xi, \eta \in y_1)[\xi \leq \eta \rightarrow f_2(\eta) \setminus f_2(\xi) \subseteq n_1]$,
- (iv) $(\forall \xi \in x_1)(\forall \eta \in y_1)[(a_\xi \cap n_2) \setminus f_2(\eta) \subseteq n_1]$.

$(\mathbb{P}_{a,\kappa}, \leq)$ is a partial ordering intended to add a \subset^* -decreasing κ -sequence $b = \langle b_\eta : \eta < \kappa \rangle$ on top of a . If G is $\mathbb{P}_{a,\kappa}$ -generic then

$$b_\eta = \bigcup \{f_p(\eta) : (\exists p \in G)[p = \langle x_p, y_p, n_p, f_p \rangle \wedge \eta \in y_p]\}.$$

The fact that b is a \subset^* -decreasing κ -sequence follows from (iii) and that for each $\xi < \eta < \kappa$ and each $m < \omega$ the sets

$$E_{\xi,\eta,m} = \{\langle x, y, n, f \rangle \in \mathbb{P}_{a,\kappa} : \xi, \eta \in y \wedge f(\xi) \setminus f(\eta) \not\subseteq m\}$$

are dense in $\mathbb{P}_{a,\kappa}$. Each b_η has a name in the ground model which is denoted by \dot{b}_η .

Lemma 2.2. $(\mathbb{P}_{a,\kappa}, \leq)$ has the ccc.

Proof. Let $A = \{p_\alpha : \alpha < \omega_1\} \subseteq \mathbb{P}_{a,\kappa}$ with $p_\alpha = \langle x_\alpha, y_\alpha, n_\alpha, f_\alpha \rangle$. Without loss of generality assume the following:

- (1) $(\forall \alpha < \omega_1)[n_\alpha = n]$ for some fixed $n < \omega$,
- (2) $\{y_\alpha : \alpha < \omega_1\}$ forms a Δ -system with root r (by the Δ -system Lemma),
- (3) $(\forall \alpha, \beta < \omega_1)[f_\alpha \upharpoonright r = f_\beta \upharpoonright r]$.

If $\alpha, \beta < \omega_1$, define $p = \langle x, y, n, f \rangle$ as follows:

$$x = x_\alpha \cup x_\beta, y = y_\alpha \cup y_\beta,$$

$$(\forall \xi) \left[(\xi \in y_\alpha \rightarrow f(\xi) = f_\alpha(\xi)) \wedge (\xi \in y_\beta \rightarrow f(\xi) = f_\beta(\xi)) \right].$$

Condition (3) implies that f is well defined. With p defined in such a way it follows that $p \in \mathbb{P}_{a,\kappa}$ and $p \leq p_\alpha, p_\beta$ so that A cannot be an antichain. \square

In fact, in the proof above it can be seen that any finite number of elements of A are compatible so that $\mathbb{P}_{a,\kappa}$ has precaliber ω_1 . It is easily seen that if κ is countable then $\mathbb{P}_{a,\kappa}$ is σ -centered. This fact will be needed in Section 3.

The next lemma is taken from [6] and is used in the proof of the proposition below.

Lemma 2.3. Assume M is a countable transitive model (c.t.m.) of ZFC, $\mathbb{P} \in M$, $E \subseteq \mathbb{P}$ and $E \in M$. Let G be \mathbb{P} -generic over M . Then if $p \in G$ and E is dense below p , then $G \cap E \neq \emptyset$.

Proposition 2.4. Let M be a c.t.m. for ZFC and in M , assume that λ is regular and κ a regular uncountable cardinal. Also assume that in M , $a = \langle a_\xi : \xi < \lambda \rangle$ is a \subset^* -increasing λ -sequence in $\mathcal{P}(\omega)$. If G is $\mathbb{P}_{a,\kappa}$ -generic over M , then in $M[G]$, $\langle a, b \rangle$ is a strong (λ, κ) -gap with $b = \langle b_\eta : \eta < \kappa \rangle$ as defined from G earlier.

Proof. By way of contradiction assume that $\langle a, b \rangle$ fails to be a strong gap in $M[G]$. Chose a nice name τ for a subset of ω and $p_0 \in G$ such that

$$p_0 \Vdash (\forall \check{\xi} < \check{\lambda})(\forall \check{\eta} < \check{\kappa})[|\tau \setminus \check{a}_{\check{\xi}}| = \check{\omega} \wedge |\tau \setminus \check{b}_{\check{\eta}}| < \check{\omega}].$$

Then $\tau = \bigcup \{ \{\check{n}\} \times A_n : n < \omega \}$ where each A_n is a countable antichain in $\mathbb{P}_{a,\kappa}$. Let $\alpha < \kappa$ be large enough so that $\bigcup_{n < \omega} A_n \subseteq \mathbb{P}_{a,\alpha}$. Fix $\beta < \kappa$ with $\alpha < \beta$ and for each $m < \omega$ let

$$F_{\beta,m} = \left\{ p \in \mathbb{P}_{a,\kappa} : \beta \in y_p \wedge (\exists i < \omega) \left[m < i \wedge p \Vdash (\check{i} \in \tau \setminus \check{b}_{\beta}) \right] \right\}.$$

Then $F_{\beta,m} \in M$ is dense below p_0 in $\mathbb{P}_{a,\kappa}$. To show the denseness part, let $q \leq p_0$. Without loss of generality assume that $\beta \in y_q$. Then

$$q \Vdash (\forall \check{\xi} < \check{\lambda})(\forall \check{\eta} < \check{\kappa})[|\tau \setminus \check{a}_{\check{\xi}}| = \check{\omega} \wedge |\tau \setminus \check{b}_{\check{\eta}}| < \check{\omega}].$$

Since x_q is finite it follows that $q \Vdash (|\tau \setminus \bigcup_{\xi \in x_q} \check{a}_{\xi}| = \check{\omega})$. Fix $n > \max(m, n_q)$. Then $q \Vdash (\exists i > n)[\check{i} \in (\tau \setminus \bigcup_{\xi \in x_q} \check{a}_{\xi})]$. Fix $r \leq q$ and $i > n$ with $r \Vdash (\check{i} \in (\tau \setminus \bigcup_{\xi \in x_q} \check{a}_{\xi}))$. Then r is compatible with some element of A_i . In fact, without loss of generality, assume that r actually extends some element of A_i . Further assume that $\max(f_r(\beta)) > i$. Now define $p = \langle x_p, y_p, n_p, f_p \rangle$ as follows

$$x_p = x_r, y_p = y_r, n_p = n_r, \\ (\forall \xi \in y_p) [(\xi < \beta \rightarrow f_p(\xi) = f_r(\xi)) \wedge (\xi \geq \beta \rightarrow f_p(\xi) = f_r(\xi) \setminus \{i\})].$$

Then clearly $p \in \mathbb{P}_{a,\kappa}$, $p \Vdash (\check{i} \notin \check{b}_{\beta})$ and p still extends some element of A_i (by the choice of β). Thus $p \Vdash (\check{i} \in \tau \setminus \check{b}_{\beta})$ and $p \in F_{\beta,m}$ with $p \leq q$. This shows that $F_{\beta,m}$ is dense below p_0 in $\mathbb{P}_{a,\kappa}$. But G is $\mathbb{P}_{a,\kappa}$ -generic over M and $p_0 \in G$, so by Lemma 2.3, $G \cap F_{\beta,m} \neq \emptyset$ for each $m < \omega$. Hence $|\tau_G \setminus b_{\beta}| = \omega$. Contradiction. \square

The proof of the main result is by iterated ccc forcing construction with finite support. The next set of results is intended to show that the strong gaps already constructed do not all of a sudden disappear at later successor stages of the construction.

Lemma 2.5. *Let M be a c.t.m. for ZFC and in M , assume λ is regular and κ a regular uncountable cardinal with $\lambda < \kappa$. Also suppose that in M , $\langle a, b \rangle$ is a strong (λ, κ) -gap and \mathbb{P} a ccc partial order with $|\mathbb{P}| < \kappa$. If G is \mathbb{P} -generic over M then $\langle a, b \rangle$ is a strong (λ, κ) -gap in $M[G]$.*

Proof. By way of contradiction, assume that $\tau \in M^{\mathbb{P}}$ is such that in $M[G]$, $(\forall \xi < \lambda)(\forall \eta < \kappa)[|\tau_G \setminus a_{\xi}| = \omega \wedge |\tau_G \setminus b_{\eta}| < \omega]$. Then $(\exists n < \omega)(\exists Y \in [\kappa]^{\kappa} \cap M[G])(\forall \eta \in Y)[\tau_G \setminus b_{\eta} \subseteq n]$. Fix one such n and Y . Then each $\eta \in Y$ is forced to be there by some $p \in G$. But $G \subseteq \mathbb{P}$ so $|G| \leq |\mathbb{P}| < \kappa$. Hence for some $p_0 \in G$, $X = \{\eta \in \kappa : p_0 \Vdash (\check{\eta} \in \check{Y})\}$ has cardinality κ . Furthermore $X \in M$. Consequently $\tau_G \subseteq^* \bigcap_{\eta \in X} b_{\eta}$ and $\bigcap_{\eta \in X} b_{\eta} \in M$ so that $\bigcap_{\eta \in X} b_{\eta}$ witnesses that $\langle a, b \rangle$ is not strong in M , a contradiction. Hence $\langle a, b \rangle$ is a strong (λ, κ) -gap in $M[G]$. \square

Definition 2.6. Let \mathbb{P} and \mathbb{Q} be partial orders. An $i: \mathbb{P} \rightarrow \mathbb{Q}$ is a complete embedding if

- (1) $(\forall p, p' \in \mathbb{P})[p' \leq p \rightarrow i(p') \leq i(p)],$
- (2) $(\forall p, p' \in \mathbb{P})[p' \perp p \leftrightarrow i(p') \perp i(p)],$
- (3) $(\forall q \in \mathbb{Q})(\exists p \in \mathbb{P})(\forall p' \in \mathbb{P})[p' \leq p \rightarrow i(p') \perp q].$

The following lemma is taken from [6].

Lemma 2.7. Suppose $i, \mathbb{P}, \mathbb{Q}$ are in M , $i: \mathbb{P} \rightarrow \mathbb{Q}$ and i is a complete embedding. Let H be \mathbb{Q} -generic over M . Then $i^{-1}(H)$ is \mathbb{P} -generic over M and $M[i^{-1}(H)] \subseteq M[H]$.

Let κ be an ordinal and $X \subseteq \kappa$. Then it is easily seen that $\mathbb{P}_{a,X} \subseteq \mathbb{P}_{a,\kappa}$. Let $i: \mathbb{P}_{a,X} \rightarrow \mathbb{P}_{a,\kappa}$ be the inclusion map defined so that if $p \in \mathbb{P}_{a,X}$ then $i(p) = p$.

Lemma 2.8. Let λ, κ be ordinals, $X \subseteq \kappa$ and $\langle a_\xi: \xi < \lambda \rangle$ a \subset^* -increasing λ -sequence in $\mathcal{P}(\omega)$. Then the inclusion $i: \mathbb{P}_{a,X} \rightarrow \mathbb{P}_{a,\kappa}$ is a complete embedding.

Proof. Properties (1) and (2) of Definition 2.6 are trivial. For (3), let $q = \langle x_q, y_q, n_q, f_q \rangle \in \mathbb{P}_{a,\kappa}$. Then $p = \langle x_q, y_q \cap X, n_q, f_q \upharpoonright (y_q \cap X) \rangle$ has the required property. \square

Proposition 2.9. Let M be a c.t.m. for ZFC and in M let λ, μ, κ be regular cardinals with $\omega, \lambda, \mu < \kappa$. Also assume that in M , $\langle c, d \rangle$ is a strong (μ, κ) -gap and $a = \langle a_\xi: \xi < \lambda \rangle$ a \subset^* -increasing λ -sequence in $\mathcal{P}(\omega)$. If G is $\mathbb{P}_{a,\kappa}$ -generic over M , then in $M[G]$, $\langle c, d \rangle$ remains a strong gap.

Proof. Let $\langle c, d \rangle = \langle c_\xi, d_\eta: \xi < \mu, \eta < \kappa \rangle$ and by way of contradiction suppose that $\tau \in M^{\mathbb{P}_{a,\kappa}}$ is a nice name for a subset of ω such that in $M[G]$,

$$(\forall \xi < \mu)(\forall \eta < \kappa)[|\tau_G \setminus c_\xi| = \omega \wedge |\tau_G \setminus d_\eta| < \omega].$$

Then $\tau = \bigcup \{ \{\check{n}\} \times A_n : n < \omega \}$ where each A_n is a countable antichain in $\mathbb{P}_{a,\kappa}$. Let

$$X = \bigcup \{ y : n < \omega \wedge (\exists p \in A_n)[p = \langle x_p, y_p, n_p, f_p \rangle \wedge y = y_p] \}.$$

Then $\mathbb{P}_{a,X}$ is a complete suborder of $\mathbb{P}_{a,\kappa}$ and $G_X = G \cap \mathbb{P}_{a,X}$ is $\mathbb{P}_{a,X}$ -generic over M with $M[G_X] \subseteq M[G]$. This follows from the previous two lemmas. Furthermore,

$$\tau_G = \{ n : (\exists p \in G)[p \in A_n] \} = \{ n : (\exists p \in G_X)[p \in A_n] \} = \tau_{G_X} \in M[G_X].$$

Therefore $\langle c, d \rangle$ fails to be strong in $M[G_X]$. But this contradicts Lemma 2.5 since X is countable, $\lambda < \kappa$, so $|\mathbb{P}_{a,X}| < \kappa$. Hence $\langle c, d \rangle$ remains a strong (μ, κ) -gap in $M[G]$. \square

This proposition basically shows that I can go beyond the successor stages in the iterated forcing construction. Now I need to show that I can also go beyond the limit stages. This is the content of the next two results.

Lemma 2.10. *Suppose that in M , γ is a limit ordinal of uncountable cofinality and \mathbb{P}_ξ , $\xi < \gamma$, is a sequence of ccc partial orders constructed by a finite support iteration. Let $c \subseteq \omega$ with $c \in M^{\mathbb{P}_\gamma}$. Then for some $\alpha < \gamma$, $c \in M^{\mathbb{P}_\alpha}$.*

This is a well-known result on forcing and I omit the proof.

Proposition 2.11. *Suppose that λ and κ are regular cardinals, with κ uncountable, and $\langle a, b \rangle$ a strong (λ, κ) -gap. Let \mathbb{P}_ξ , for $\xi < \gamma$, be a sequence of ccc partial orders constructed by a finite support iteration, where γ is a limit ordinal. Suppose that for each $\xi < \gamma$,*

$$1 \Vdash_{\mathbb{P}_\xi} \text{“}\langle a, b \rangle \text{ is a strong gap”}.$$

Then the same is true for $\xi = \gamma$.

Proof. There are two cases depending on whether or not $\text{cf}(\gamma) = \omega$.

Suppose $\text{cf}(\gamma) = \omega$ and that the conclusion of the proposition is false. Choose a name $\tau \in M^{\mathbb{P}_\gamma}$ such that if G is \mathbb{P}_γ -generic over M , then in $M[G]$,

$$(\forall \xi < \lambda)(\forall \eta < \kappa) [|\tau_G \setminus a_\xi| = \omega \wedge |\tau_G \setminus b_\eta| < \omega].$$

Then

$$(\exists n < \omega)(\exists X \in [\kappa]^\kappa \cap M[G])(\forall \eta \in X) [\tau_G \setminus b_\eta \subseteq n].$$

Fix one such n and X . Then each $\eta \in X$ is forced to be there by some $p_\eta \in G$. But $\langle \mathbb{P}_\xi: \xi \leq \gamma \rangle$ is a sequence of ccc partial orders constructed by a finite support iteration and κ is uncountable. Therefore

$$(\exists \xi < \gamma)(\exists Y \in [X]^\kappa)(\forall \eta \in Y) [\text{supp}(p_\eta) \subseteq \xi].$$

But this in turn implies that

$$(\exists \xi < \gamma)(\exists Y \in [\kappa]^\kappa \cap M^{\mathbb{P}_\xi}) \\ [\langle a, b \rangle \text{ fails to be a strong gap via } \bigcap_{\eta \in Y} b_\eta \text{ in } M^{\mathbb{P}_\xi}]$$

which clearly contradicts the assumption.

If $\text{cf}(\gamma) \geq \omega_1$, then Lemma 2.10 is used to show that no subsets of ω are added at stage γ , so that $\langle a, b \rangle$ remains a strong gap in $M^{\mathbb{P}_\gamma}$. \square

Finally let $\text{MA}(< \kappa)$ be the statement that Martin's axiom holds below κ . With this in mind I am ready for the statement and the proof of the main result of this section.

Theorem 2.12. *Let M be a c.t.m. for ZFC and in M , assume κ and μ are regular uncountable cardinals with $\kappa < \mu$ and $2^{<\mu} = \mu$. Then there is a generic extension of M which is a model for the following statement:*

$$\text{ZFC} + \text{MA}(< \kappa) + \text{sg}(< \kappa, \kappa) + \mathfrak{t} = \kappa + \mathfrak{b} = \kappa + 2^\omega = \mu.$$

Proof. Start with a c.t.m. M for ZFC such that in M , κ and μ are regular with $\omega < \kappa < \mu$, $2^{<\mu} = \mu$ and $2^\omega = \mu$. Now perform a ccc iterated forcing construction with finite support of length μ . At even stages make sure that $\text{MA}(<\kappa)$ holds and at odd stages make sure that $\text{sg}(<\kappa, \kappa)$ holds. Use the fact that $2^{<\mu} = \mu$ in M so that all the \subset^* -increasing sequences of size $<\kappa$ in $\mathcal{P}(\omega)$ are covered by the construction. Eventually I end up with a ccc partial order \mathbb{P}_μ of size μ . In $M^{\mathbb{P}_\mu}$ it is easily seen that $\text{MA}(<\kappa)$ and $\text{sg}(<\kappa, \kappa)$ hold. Furthermore, since $|\mathbb{P}_\mu| = \mu$, I also have that $2^\omega = \mu$ in $M^{\mathbb{P}_\mu}$. In addition $\text{sg}(<\kappa, \kappa)$ implies that $\mathfrak{t} \geq \kappa$. But $\text{sg}(<\kappa, \kappa)$ also implies that there are $(1, \kappa)$ -gaps so that $\mathfrak{t} = \kappa$. Finally, $\text{MA}(<\kappa)$ implies that $\mathfrak{b} \geq \kappa$ and $\text{sg}(<\kappa, \kappa)$ implies that there are (ω, κ) -gaps so that $\mathfrak{b} = \kappa$. This finishes the proof. \square

3.

In order to preserve strong gaps, in the previous construction, I had to assume that $\lambda < \kappa$. But, what happens when $\lambda = \kappa$? This is the consideration of this section. Theorems 3.3 and 3.4 are the main results and they can be seen as refinements of Hausdorff's theorem on (ω_1, ω_1) -gaps when other assumptions are made in addition to ZFC. But first some preliminary results.

Definition 3.1. Let $\langle a, b \rangle = \langle a_\xi, b_\eta; \xi < \lambda, \eta < \kappa \rangle$ be a (λ, κ) -pre-gap where λ and κ are ordinals.

$$\mathbb{S}_{a,b} = \left\{ \langle x, y, n, f \rangle : x \in [\lambda]^{<\omega} \wedge y \in [\kappa]^{<\omega} \wedge n < \omega \wedge (f : n \rightarrow 2) \right. \\ \left. \wedge \left(\bigcup_{\xi \in x} a_\xi \setminus n \subseteq \bigcap_{\eta \in y} b_\eta \right) \right\}$$

with $\langle x_2, y_2, n_2, f_2 \rangle \leq \langle x_1, y_1, n_1, f_1 \rangle$ iff

- (1) $x_1 \subseteq x_2, y_1 \subseteq y_2, n_1 \leq n_2, f_2 \upharpoonright n_1 = f_1$,
- (2) $(\forall i < \omega)[n_1 \leq i < n_2 \rightarrow ((i \in \bigcup_{\xi \in x_1} a_\xi \rightarrow f_2(i) = 1) \wedge (i \notin \bigcap_{\eta \in y_1} b_\eta \rightarrow f_2(i) = 0))]$.

$(\mathbb{S}_{a,b}, \leq)$ is easily seen to be a partial order intended to split $\langle a, b \rangle$. The splitting set is $c = \{i : (\exists \langle x, y, n, f \rangle \in G)[f(i) = 1]\}$, where G is $\mathbb{S}_{a,b}$ -generic. This partial order is due to Kunen as is the following lemma which can also be found in [1].

Lemma 3.2. Let $\langle a, b \rangle$ be a (λ, κ) -pre-gap.

- (i) If the pre-gap is split then $\mathbb{S}_{a,b}$ has the ccc.
- (ii) If $\text{cf}(\lambda) \neq \omega_1$ or $\text{cf}(\kappa) \neq \omega_1$ then $\mathbb{S}_{a,b}$ has the ccc.
- (iii) If $\lambda = \kappa = \omega_1$ and $\langle a, b \rangle$ is a gap, then there is a ccc partial order \mathbb{Q} which adjoins an uncountable antichain in $\mathbb{S}_{a,b}$.

Theorem 3.3. $\text{MA}(\omega_1) \rightarrow g(\omega_1, \omega_1)$.

Proof. Let M be a c.t.m. for $ZFC + MA(\omega_1)$. Also in M , let $a = \langle a_\xi : \xi < \omega_1 \rangle$ be a \subset^* -increasing ω_1 -sequence in $\mathcal{P}(\omega)$. \mathbb{P}_{a, ω_1} is the partial order which adjoins a \subset^* -decreasing ω_1 -sequence $b = \langle b_\eta : \eta < \omega_1 \rangle$ on top of a . Let G_0 be \mathbb{P}_{a, ω_1} -generic over M . Then by Proposition 2.4, $\langle a, b \rangle$ is an (ω_1, ω_1) -gap in $M[G_0]$. Let \mathbb{Q} be the ccc partial order as in (iii) of Lemma 3.2. So $1 \Vdash_{\mathbb{P}_{a, \omega_1}} (\dot{\mathbb{Q}} \text{ has the ccc})$. Hence, the two-step iterated forcing construction $\mathbb{P}_{a, \omega_1} * \dot{\mathbb{Q}}$ has the ccc in M . Now, if G is $\mathbb{P}_{a, \omega_1} * \dot{\mathbb{Q}}$ -generic over M , then in $M[G]$ there exists an uncountable antichain A in $\mathbb{S}_{a, b}$. But the sequence b is obtained from the intersection of G and ω_1 dense sets of $\mathbb{P}_{a, \omega_1} * \dot{\mathbb{Q}}$ which lie in M . In addition, the uncountable antichain A is also obtained from the intersection of G and ω_1 dense sets of $\mathbb{P}_{a, \omega_1} * \dot{\mathbb{Q}}$ which also lie in M . So, there are ω_1 dense sets of $\mathbb{P}_{a, \omega_1} * \dot{\mathbb{Q}}$ in M which decide both A and b . But by $MA(\omega_1)$ there is a filter in M which intersects all these ω_1 dense sets. Therefore there is a sequence b and an uncountable antichain of $\mathbb{S}_{a, b}$ which are in M . Hence by Lemma 3.2, $\langle a, b \rangle$ is an (ω_1, ω_1) -gap in M . \square

Because of Lemma 3.2 this result cannot be generalized to higher cardinals. However there is the following

Theorem 3.4. *Let M be a c.t.m. for ZFC and in M , assume that κ and μ are regular uncountable cardinals with $\kappa < \mu$ and $2^{<\mu} = \mu$. Then there is a generic extension of M which is a model for $ZFC + MA(<\kappa) + g(\kappa, \kappa) + 2^\omega = \mu$.*

In order to prove this theorem I need

Lemma 3.5. *If κ is a regular uncountable cardinal, \mathbb{P} a σ -centered partial order, and $\langle a, b \rangle$ a (κ, κ) -gap, then \mathbb{P} cannot split $\langle a, b \rangle$.*

Proof. Let M be a c.t.m. for ZFC and in M , let κ be a regular uncountable cardinal, $\langle a, b \rangle$ a (κ, κ) -gap and \mathbb{P} a σ -centered partial order. By way of contradiction assume that τ is a \mathbb{P} -name and $p \in \mathbb{P}$ such that $p \Vdash \tau \text{ splits } \langle a, b \rangle$. For each $\xi < \kappa$ choose $p_\xi \leq p$ and $n_\xi < \omega$ with $p_\xi \Vdash (\check{a}_\xi \setminus \check{n}_\xi \subseteq \tau \setminus \check{n}_\xi \subseteq \check{b}_\xi)$. Since κ is uncountable, without loss of generality, assume that $(\forall \xi < \kappa)[n_\xi = n]$ for some $n < \omega$. But $\mathbb{P} = \bigcup_{n < \omega} \mathbb{P}_n$ where each \mathbb{P}_n is centered. So choose $X \in [\kappa]^\kappa$ and $m < \omega$ such that $(\forall \xi \in X)[p_\xi \in \mathbb{P}_m]$. Hence $(\forall \xi, \eta \in X)[p_\xi \not\perp p_\eta]$. But this implies that $\bigcup_{\xi \in X} a_\xi$ splits $\langle a, b \rangle$ in M . Contradiction. \square

In fact, it is possible to show that σ -centered partial orders cannot destroy (λ, κ) -gaps for $\omega_1 \leq \lambda \leq \kappa$. So that in Theorem 3.4, $g(\kappa, \kappa)$ can be replaced by $g(\lambda, \kappa)$ for $\omega_1 \leq \lambda \leq \kappa$. But the proof is more complicated and I choose not to present it here.

Proof of Theorem 3.4. Let M be a c.t.m. for ZFC such that in M , κ and μ are regular uncountable cardinals, $\kappa < \mu$, $2^{<\mu} = \mu$ and $2^\omega = \mu$. The proof is by ccc iterated forcing construction with finite support of length μ . At even stages

consider ccc partial orders of size $< \kappa$ so that $\text{MA}(< \kappa)$ holds in $M^{\mathbb{P}_\mu}$. Lemma 2.5 shows that previously constructed (κ, κ) -gaps are not destroyed at such stages. At odd stages extend with $\mathbb{P}_{a, \kappa}$ where a is a \subset^* -increasing κ -sequence so that eventually $g(\kappa, \kappa)$ holds. Lemma 3.5 implies that $\mathbb{P}_{a, \kappa}$ does not destroy any previously constructed (κ, κ) -gaps, since any nice name for a subset of ω is contained in a complete σ -centered suborder of $\mathbb{P}_{a, \kappa}$. Limit stages are treated in the same way as in Theorem 2.12. Finally $M^{\mathbb{P}_\mu}$ has the desired properties. \square

The results of Theorems 3.3 and 3.4 cannot be improved to strong gaps since $g(\kappa, \kappa)$ implies that there are no strong (κ, κ) -gaps. To see this let $\langle a_\xi, b_\xi: \xi < \kappa \rangle$ be a (κ, κ) -gap, and let $b_\xi^c = \omega \setminus b_\xi$. Then $\langle a_\xi \cup b_\xi^c: \xi < \kappa \rangle$ is a \subset^* -increasing κ -sequence and by $g(\kappa, \kappa)$ there is a \subset^* -decreasing κ -sequence $\langle d_\xi: \xi < \kappa \rangle$ such that $\langle a_\xi \cup b_\xi^c, d_\xi: \xi < \kappa \rangle$ is a (κ, κ) -gap. Let $d = d_0 \setminus d_1$ and note that d is infinite with $(\forall \xi < \kappa)[d \perp (a_\xi \cup b_\xi^c)]$. But then $(\forall \xi < \kappa)[d \perp a_\xi \wedge d \subset^* b_\xi]$ so that d witnesses that $\langle a_\xi, b_\xi: \xi < \kappa \rangle$ cannot be a strong (κ, κ) -gap.

4.

In this section, a gap is represented by two \subset^* -increasing sequences such that any element in one sequence is almost disjoint from any element in the other sequence, and no subset of ω almost contains all the elements in one sequence and is almost disjoint from all the elements in the other sequence. A tight gap is a gap with no infinite subset of ω almost disjoint from either of the sequences.

In connection with the existence of a separable, countably compact, noncompact manifold in all models of $\mathfrak{p} = \omega_1$, Nyikos asked whether $\mathfrak{p} = \omega_1$ implies the existence of a tight (ω_1, ω_1) -gap in ${}^\omega\omega$. He defines a tight gap in ${}^\omega\omega$ as follows:

Definition 4.1. Given a function $f \in {}^\omega\omega$, define

$$f^\uparrow = \{\langle i, j \rangle \in \omega \times \omega: j \geq f(i)\},$$

$$f^\downarrow = \{\langle i, j \rangle \in \omega \times \omega: j \leq f(i)\}.$$

Call a pair $\langle a, b \rangle$ of families in ${}^\omega\omega$ a tight (λ, κ) -gap if $a = \langle f_\xi: \xi < \lambda \rangle$, $b = \langle g_\eta: \eta < \kappa \rangle$ are such that $\langle f_\xi^\downarrow, g_\eta^\uparrow: \xi < \lambda, \eta < \kappa \rangle$ is a tight (λ, κ) -gap in $\omega \times \omega$.

According to this definition, a positive answer follows almost trivially from Theorem 4.2 below. Its proof is a modification of the proof by van Douwen in [12] of the existence of (ω_1, ω_1) -gaps which in turn is decodable from Hausdorff's original proof in [3,4]. This result was independently discovered by Nyikos and Vaughan in [8] and the implication from left to right is a part of a more general result by Błaszczyk and Szymanski in [2].

Theorem 4.2. $\mathfrak{p} = \omega_1$ iff there is a tight (ω_1, ω_1) -gap in $\mathcal{P}(\omega)$.

Proof. The existence of a tight (ω_1, ω_1) -gap $\langle a_\xi, b_\eta; \xi, \eta < \omega_1 \rangle$ in $\mathcal{P}(\omega)$ implies that $t = \omega_1$, hence $p = \omega_1$, since $\omega_1 \leq p \leq t$ and by considering the sequence $\langle a_\xi \cup b_\xi; \xi < \omega_1 \rangle$.

Now assume that $p = \omega_1$. Since $p = \omega_1 \rightarrow t = \omega_1$ (as shown by Rothberger in [11]), let $\mathcal{T} = \langle T_\xi; \xi < \omega_1 \rangle$ be a \subset^* -well-ordered sequence of subsets of ω , in type ω_1 , such that no infinite subset of ω is almost disjoint from all T_ξ . The construction of a tight (ω_1, ω_1) -gap in $\mathcal{P}(\omega)$ is by induction. Two families $\langle a_\xi; \xi < \omega_1 \rangle$ and $\langle b_\eta; \eta < \omega_1 \rangle$ are constructed with the following properties:

- (1) $(\forall \xi < \omega_1)[a_\xi \cup b_\xi = T_\xi]$,
- (2) $(\forall \xi < \omega_1)[a_\xi \cap b_\xi = \emptyset]$,
- (3) $(\forall \xi, \eta < \omega_1)[\xi < \eta \rightarrow a_\xi \subset^* a_\eta \wedge b_\xi \subset^* b_\eta]$,
- (4) $(\forall \eta < \omega_1)(\forall k < \omega)[|\{\xi < \eta: a_\eta \cap b_\xi \subseteq k\}| < \omega]$.

If $\xi = 0$, split T_0 into two infinite disjoint sets and let a_0 be one of them and b_0 be the other.

If ξ is a successor with $\xi = \zeta + 1$, split $T_\xi \setminus T_\zeta$ into two infinite disjoint sets c and d and let $a_\xi = a_\zeta \cup c$ and $b_\xi = b_\zeta \cup d$.

Now assume that ξ is a limit ordinal. Since ξ is countable and $\text{MA}(\omega)$ is always true it follows by Lemma 3.2 that there is an $S \in [\omega]^\omega$ such that $(\forall \zeta < \xi)[a_\zeta \subset^* S \wedge b_\zeta \perp S]$. Since $(\forall \zeta < \xi)[a_\zeta \cup b_\zeta \subset^* T_\xi]$ by the induction hypothesis I may assume that $S \subseteq T_\xi$. For each $n < \omega$ let $X_n = \{\zeta < \xi: S \cap b_\zeta \subseteq n\}$. For $\Sigma \subseteq \omega$ and $\Gamma \subseteq \xi$ let $\Sigma(\text{ct})\Gamma$ abbreviate that $\{\zeta \in \Gamma: \Sigma \cap b_\zeta \subseteq m\}$ is finite for each $m < \omega$.

By recursion I construct a sequence $\langle S_n; n < \omega \rangle$ of subsets of T_ξ satisfying $S_0 = S$ and $(\forall \zeta < \xi)[S_n \perp b_\zeta]$ and $S_n \subseteq S_{n+1}$ and $S_{n+1}(\text{ct})X_n$ for $n < \omega$.

Let $n < \omega$ and suppose S_n is known. If X_n is finite let $S_{n+1} = S_n$. So suppose X_n is infinite. Then by the choice of S and the induction hypothesis it follows that $(\forall \zeta < \xi)[|X_n \cap \zeta| < \omega]$. Hence X_n is cofinal in ξ and has order type ω . Let $e: \omega \rightarrow X_n$ be a strictly increasing surjection. Then for each $n < \omega$, $T_\xi \cap (b_{e(n)} \setminus \bigcup_{i < n} b_{e(i)})$ is infinite, so pick $j(n)$ in it with $j(n) \geq n$. Clearly $(\forall \zeta \in X_n)[\text{ran}(j) \perp b_\zeta]$, hence $(\forall \zeta < \xi)[\text{ran}(j) \perp b_\zeta]$ since X_n is cofinal in ξ . Let $S_{n+1} = S_n \cup \text{ran}(j)$ and note that since $(\forall \zeta < \xi)[S_n \perp b_\zeta]$ the same is true for S_{n+1} . Furthermore, $\text{ran}(j)(\text{ct})X_n$ by construction, hence $S_{n+1}(\text{ct})X_n$. This completes the construction of $\langle S_n; n < \omega \rangle$.

Using $\text{MA}(\omega)$ and Lemma 3.2 once again, there is an $a_\xi \subset T_\xi$ with $a_\xi \perp b_\zeta$ for each $\zeta < \xi$ and $S_n \subset^* a_\xi$ for each $n < \omega$. In addition, since $S \subset T_\xi$ I may assume that $S \subset a_\xi$. Let $b_\xi = T_\xi \setminus a_\xi$.

Clearly (1), (2), and (3) are satisfied for ξ . To prove (4) for ξ , suppose that there is an $m < \omega$ such that $X = \{\zeta < \xi: a_\xi \cap b_\zeta \subseteq m\}$ is infinite. As $S \subseteq a_\xi$, it follows that $X \cap X_n$ is infinite for some $n \leq m$. But $S_{n+1}(\text{ct})X_n$ hence it follows from $S_{n+1} \subset^* a_\xi$ that $a_\xi(\text{ct})X_n$ which leads to the absurdity that $X \cap X_n$ is finite. This finishes the construction of $\langle a_\xi; \xi < \omega_1 \rangle$ and $\langle b_\eta; \eta < \omega_1 \rangle$.

Now I show that $\langle a_\xi, b_\eta; \xi, \eta < \omega_1 \rangle$ forms a tight (ω_1, ω_1) -gap in $\mathcal{P}(\omega)$. Tightness follows from (1) and the fact that no infinite subset of ω is almost disjoint from all T_ξ . To show that $\langle a_\xi, b_\eta; \xi, \eta < \omega_1 \rangle$ is a gap, by way of contradic-

tion, assume that $(\exists d \in [\omega]^\omega)(\forall \xi, \eta < \omega_1)[a_\xi \subset^* d \wedge b_\eta \perp d]$. Then $(\exists \Sigma \in [\omega_1]^{\omega_1})(\exists n < \omega)(\forall \xi \in \Sigma)(a_\xi \setminus d \subset n)$ and $(\exists \Gamma \in [\Sigma]^{\omega_1})(\exists m < \omega)(\forall \eta \in \Gamma)(d \cap b_\eta \subseteq m)$. Choose $\xi < \omega_1$ big enough so that $\xi \cap \Gamma$ is infinite. Then $\{\zeta < \xi: a_\xi \cap b_\zeta \subseteq \max(m, n)\}$ is infinite. This contradicts (4). So $\langle a_\xi, b_\eta: \xi, \eta < \omega_1 \rangle$ is a tight (ω_1, ω_1) -gap in $\mathcal{P}(\omega)$. \square

Corollary 4.3. $\mathfrak{p} = \omega_1$ iff there is a tight (ω_1, ω_1) -gap in ${}^\omega\omega$.

Proof. The existence of a tight (ω_1, ω_1) -gap in ${}^\omega\omega$ clearly implies that $\mathfrak{p} = \omega_1$. So assume that $\mathfrak{p} = \omega_1$ and let $\langle a_\xi, b_\eta: \xi, \eta < \omega_1 \rangle$ be a tight (ω_1, ω_1) -gap in $\mathcal{P}(\omega)$. Let $f_\xi(n) = \chi_{a_\xi}(n)$ and $g_\eta(n) = 2 - \chi_{b_\eta}(n)$ where χ_A is a characteristic function of $A \subseteq \omega$.

I claim that $\langle f_\xi^\downarrow, g_\eta^\uparrow: \xi, \eta < \omega_1 \rangle$ is a tight (ω_1, ω_1) -gap in ${}^\omega\omega$. If $c \in [\omega \times \omega]^\omega$ splits $\langle f_\xi^\downarrow, g_\eta^\uparrow: \xi, \eta < \omega_1 \rangle$, then $\{n: (n, 1) \in c \cap (\omega \times \{1\})\}$ clearly splits $\langle a_\xi, b_\eta: \xi, \eta < \omega_1 \rangle$. And if $d \in [\omega \times \omega]^\omega$ destroys the tightness of $\langle f_\xi^\downarrow, g_\eta^\uparrow: \xi, \eta < \omega_1 \rangle$, then $\{n: (n, 1) \in d \cap (\omega \times \{1\})\}$ clearly destroys the tightness of $\langle a_\xi, b_\eta: \xi, \eta < \omega_1 \rangle$. But such c and d cannot exist since $\langle a_\xi, b_\eta: \xi, \eta < \omega_1 \rangle$ is a tight (ω_1, ω_1) -gap in $\mathcal{P}(\omega)$. \square

5. References

- [1] J.E. Baumgartner, Applications of the proper forcing axiom, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 913–959.
- [2] A. Blaszczyk and A. Szymanski, Hausdorff's gaps versus normality, Bull. Acad. Polon. Sci. Ser. Sci. Math. 30 (1982) 371–378.
- [3] F. Hausdorff, Die Graduierung nach dem Endverlauf, Abh. Königlich Sächs. Gesellschaft Wiss. Math. Phys. Kl. 31 (1909) 296–334.
- [4] F. Hausdorff, Summen von \aleph_1 Mengen, Fund. Math. 26 (1936) 241–255.
- [5] C.D. Herink, Some applications of iterated forcing, Ph.D. thesis, University of Wisconsin-Madison, Madison, WI (1977).
- [6] K. Kunen, Set Theory. An Introduction to Independence Proofs (North-Holland, Amsterdam, 1980).
- [7] P. Nyikos, On first countable, countably compact spaces III, in: J. van Mill and G.M. Reed, eds., Open Problems in Topology (North-Holland, Amsterdam, 1990) 127–161.
- [8] P. Nyikos and J.E. Vaughan, On first countable, countably compact spaces I: (ω_1, ω_1^*) -gaps, Trans. Amer. Math. Soc. 279 (1983) 463–469.
- [9] M. Rabus, Tight gaps in $\mathcal{P}(\omega)$, Topology Proc., to appear.
- [10] F. Rothberger, Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété C, Proc. Cambridge Philos. Soc. 37 (1941) 109–126.
- [11] F. Rothberger, On some problems of Hausdorff and Sierpiński, Fund. Math. 35 (1948) 29–46.
- [12] E.K. van Douwen, The integers in topology, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam 1984) 111–167.